

# Bound states of a more general exponential screened Coulomb potential

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An alternative approximation scheme has been used in solving the Schrödinger equation to the more general case of exponential screened Coulomb potential,  $V(r) = -(a/r) [1 + (1 + br)e^{-2br}]$ . The bound state energies of the 1s, 2s and 3s-states, together with the ground state wave function are obtained analytically upto the second perturbation term.

**KEY WORDS:** exponential screened Coulomb potential, perturbation theory

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## 1. Introduction

A more general exponential screened Coulomb (MGESC) potential of the form:

$$V(r) = -\left(\frac{a}{r}\right) [1 + (1 + br) \exp(-2br)], \quad (1)$$

where  $a$  is the strength coupling constant and  $b$  is the screening parameter, is known to describe adequately the effective interaction in many-body environment of a variety of fields such as atomic, nuclear, solid-state, plasma physics and quantum field theory [1,2]. It is also used in describing the potential between an ionized impurity and an electron in a metal [3,4] or a semiconductor [5] and the electron–positron interaction in a positronium atom in a solid [6].

The Schrödinger equation for such a potential does not admit exact solutions, various approximate methods both numerical [7] and analytical [8] have

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been developed. The MGESC potential defined for an electron of the helium atom in the field of other electrons and nucleus has been investigated by Gerry and Laub [9]. Further, the large- $N$  expansion was used to obtain the bound-state energy of the ground state and the first excited state and the corresponding wave functions analytically by Sever and Tezcan [10].

In this paper, we calculate the binding energy eigenvalues of MGESC potential using a novel perturbative formalism [11] which has recently been used in solving the Schrödinger equation to obtain the bound-state energies as well as the wave functions for different types of potentials [11, 12] in both bound and continuum regions. This novel treatment is based on the decomposition of the radial Schrödinger equation into two pieces having an exactly solvable part with an additional piece leading to either a closed analytical solution or approximate treatment depending on the nature of the perturbed potential.

The contents of this paper is as follows. In section 2, we briefly outline the method with all necessary formulae to perform the current calculations. In section 3, we apply this method to the Schrödinger equation using a more general exponential screening Coulomb potential to obtain analytical expressions for the bound-state energy and eigen functions of different energy states. Finally, we end with some results and conclusions in section 4.

## 2. The Method

For a spherically symmetric potential, the corresponding Schrödinger equation, in the bound state domain, with the radial wave function reads

$$\frac{\hbar^2}{2m} \frac{\psi_n''(r)}{\psi_n(r)} = V(r) - E_n \quad (2)$$

with

$$V(r) = \left[ V_0(r) + \frac{\hbar^2}{2m} \frac{\ell(\ell+1)}{r^2} \right] + \Delta V(r), \quad (3)$$

where  $\Delta V(r)$  is a perturbing potential and  $\psi_n(r) = \chi_n(r)u_n(r)$  is the full radial wave function, in which  $\chi_n(r)$  is the known normalized wave function of the unperturbed Schrödinger equation whereas  $u_n(r)$  is a moderating wave function corresponding to the perturbing potential. Following the method given in [11, 12], we may rewrite (2) as

$$\frac{\hbar^2}{2m} \left( \frac{\chi_n''(r)}{\chi_n(r)} + \frac{u_n''(r)}{u_n(r)} + 2 \frac{\chi_n'(r)u_n'(r)}{\chi_n(r)u_n(r)} \right) = V(r) - E_n. \quad (4)$$

The logarithmic derivatives of the unperturbed  $\chi_n(r)$  and perturbed  $u_n(r)$  wave functions are given by

$$W_n(r) = -\frac{\hbar}{\sqrt{2m}} \frac{\chi'_n(r)}{\chi_n(r)} \quad \text{and} \quad \Delta W_n = -\frac{\hbar}{\sqrt{2m}} \frac{u'_n(r)}{u_n(r)}, \tag{5}$$

which leads to

$$\frac{\hbar^2}{2m} \frac{\chi''_n(r)}{\chi_n(r)} = W_n^2(r) - \frac{\hbar}{\sqrt{2m}} W'_n(r) = \left[ V_0(r) + \frac{\hbar^2}{2m} \frac{\ell(\ell + 1)}{r^2} \right] - \varepsilon_n, \tag{6}$$

where  $\varepsilon_n$  is the eigenvalue for the exactly solvable potential of interest, and

$$\begin{aligned} \frac{\hbar^2}{2m} \left( \frac{u''_n(r)}{u_n(r)} + 2 \frac{\chi'_n(r)u'_n(r)}{\chi_n(r)u_n(r)} \right) &= \Delta W_n^2(r) - \frac{\hbar}{\sqrt{2m}} \Delta W'_n(r) + 2W_n(r)\Delta W_n(r) \\ &= \Delta V(r) - \Delta\varepsilon_n, \end{aligned} \tag{7}$$

in which  $\Delta\varepsilon_n = E_n^{(1)} + E_n^{(2)} + \dots$  is the correction term to the energy due to  $\Delta V(r)$  and  $E_n = \varepsilon_n + \Delta\varepsilon_n$ . If equation (7), which is the most significant piece of the present formalism, can be solved analytically as in (6), then the whole problem, in equation (2) reduces to the following form

$$[W_n(r) + \Delta W_n(r)]^2 - \frac{\hbar}{\sqrt{2m}} (W_n(r) + \Delta W_n(r))' = V(r) - E_n, \tag{8}$$

which is a well known treatment within the frame of supersymmetric quantum theory (SSQT) [13]. Thus, if the whole spectrum and corresponding eigenfunctions of the unperturbed interaction potential are known, then one can easily calculate the required superpotential  $W_n(r)$  for any state of interest leading to direct computation of related corrections to the unperturbed energy and wave function.

For the perturbation technique, we can split the given potential in equation (2) into two parts. The main part corresponds to a shape invariant potential, equation (6), for which the superpotential is known analytically and the remaining part is treated as a perturbation, equation (7). Therefore, the MGESC potential can be solved using this method. In this case, the zeroth-order term corresponds to the Coulomb potential while higher-order terms constitute the perturbation. However, the perturbation term in its present form cannot be solved exactly through equation (7). Thus, one should expand the functions related to the perturbation in terms of the perturbation parameter  $b$ ,

$$\Delta V(r; b) = \sum_{i=1}^{\infty} b_i V_i(r), \quad \Delta W_n(r; b) = \sum_{i=1}^{\infty} b_i W_n^{(i)}(r), \quad E_n^{(i)}(b) = \sum_{i=1}^{\infty} b_i E_n^{(i)}, \tag{9}$$

where  $i$  denotes the perturbation order. Substitution of the above expansions into equation (7) and equating terms with the same power of  $b$  on both sides up to  $O(b^3)$  yields

$$2W_n(r)W_n^{(1)}(r) - \frac{\hbar}{\sqrt{2m}} \frac{dW_n^{(1)}(r)}{dr} = V_1(r) - E_n^{(1)}, \quad (10)$$

$$W_n^{(1)2}(r) + 2W_n(r)W_n^{(2)}(r) - \frac{\hbar}{\sqrt{2m}} \frac{dW_n^{(2)}(r)}{dr} = V_2(r) - E_n^{(2)}, \quad (11)$$

$$2 \left[ W_n(r)W_n^{(3)}(r) + W_n^{(1)}(r)W_n^{(2)}(r) \right] - \frac{\hbar}{\sqrt{2m}} \frac{dW_n^{(3)}(r)}{dr} = V_3(r) - E_n^{(3)}. \quad (12)$$

Hence, unlike the other perturbation theories, equation (7) and its expansion, equations (10)–(12), give a flexibility for the easy calculations of the perturbative corrections to energy and wave functions for the  $n$ th state of interest through an appropriately chosen perturbed superpotential.

### 3. Application to the MGESC Potential

Considering the recent interest in various power-law potentials in the literature, we work throughout the article within the frame of low-screening parameter  $b$ . In this regard, the MGESC potential can be expanded in power series of the screening parameter  $b$  as [10, 12]

$$V(r) = -\left(\frac{a}{r}\right) [1 + (1 + br) \exp(-2br)] = -\frac{a}{r} - \frac{a}{r} \sum_{i=0}^{\infty} V_i(br)^i, \quad (13)$$

where the perturbation coefficients  $V_i$  are given by

$$V_1 = -1, \quad V_2 = 0, \quad V_3 = 2/3, \quad V_4 = -4/6, \quad V_5 = 12/30, \dots \quad (14)$$

Therefore, we apply this approximation method to the MGESC potential with the angular momentum barrier

$$V(r) = -\left(\frac{a}{r}\right) [1 + (1 + br) e^{-2br}] + \frac{\ell(\ell + 1)\hbar^2}{2mr^2} = \left[ V_0(r) + \frac{\ell(\ell + 1)\hbar^2}{2mr^2} \right] + \Delta V(r), \quad (15)$$

where the first piece is the shape invariant zeroth-order which is an exactly solvable piece corresponding to the unperturbed Coulomb potential with  $V_0(r) = -2a/r$  while  $\Delta V(r) = ab - (2ab^3/3)r^2 + (4ab^4/6)r^3 - (12ab^5/30)r^4 + \dots$  is

the perturbation term. The literature is rich with examples of particular solutions for such power-law potentials employed in different fields of physics, for recent applications see [14,15]. At this stage one may wonder why the series expansion is truncated at a lower order. This can be understood as follows. It is widely appreciated that convergence is not an important or even desirable property for series approximations in physical problems. Specifically, a slowly convergent approximation which requires many terms to achieve reasonable accuracy is much less valuable than the divergent series which gives accurate answers in a few terms. This is clearly the case for the MGESC problem [16]. However, it is worthwhile to note that the main contributions come from the first three terms. Thereby, the present calculations are performed upto the second-order involving only these additional potential terms, which suprisingly provide highly accurate results for small screening parameter  $b$ .

*3.1. Ground state calculations ( $n = 0$ )*

In the light of equation (6), the zeroth-order calculations leading to exact solutions can be carried out readily by setting the ground-state superpotential and the unperturbed exact energy as

$$\begin{aligned}
 W_{n=0}(r) &= -\frac{\hbar}{\sqrt{2m}} \frac{\ell + 1}{r} + \sqrt{2m} \frac{a}{(\ell + 1)\hbar}, \\
 E_n^{(0)} &= -\frac{2ma^2}{\hbar^2(n + \ell + 1)^2}, \quad n = 0, 1, 2, \dots
 \end{aligned}
 \tag{16}$$

and from the literature, the corresponding normalized Coulomb bound-state wave function [17]

$$\chi_n(r) = N_{n,l}^{(C)} r^{\ell+1} \exp[-\beta r] \times L_n^{2\ell+1} [2\beta r]
 \tag{17}$$

in which  $N_{n,l}^{(C)} = \left[ \frac{4ma}{(n+\ell+1)\hbar^2} \right]^{\ell+1} \frac{1}{(n+\ell+1)} \frac{1}{\sqrt{\frac{\hbar^2}{2man!} (n+2\ell+1)!}}$  is a normalized constant,  $\beta = \frac{2ma}{(n+\ell+1)\hbar^2}$  and  $L_n^k(x) = \sum_{m=0}^n (-1)^m \frac{(n+k)!}{(n-m)!(m+k)!m!} x^m$  is an associate Laguerre polynomial function [18].

For the calculation of corrections to the zeroth-order energy and wave function, one needs to consider the expressions leading to the first- and second-order perturbation given by equations (10)–(12). Multiplication of each term in these equations by  $\chi_n^2(r)$ , and bearing in mind the superpotentials given in equation (5), one can obtain the straightforward expressions for the first-order correction

to the energy and its superpotential:

$$\begin{aligned}
 E_n^{(1)} &= \int_{-\infty}^{\infty} \chi_n^2(r) \left( -\frac{2ab^3}{3} r^2 \right) dr, \\
 W_n^{(1)}(r) &= \frac{\sqrt{2m}}{\hbar} \frac{1}{\chi_n^2(r)} \int^r \chi_n^2(x) \left[ E_n^{(1)} + \frac{2ab^3}{3} x^2 \right] dx
 \end{aligned} \tag{18}$$

and also for the second-order correction and its superpotential:

$$\begin{aligned}
 E_n^{(2)} &= \int_{-\infty}^{\infty} \chi_n^2(r) \left[ \frac{4ab^4}{6} r^3 - W_n^{(1)2}(r) \right] dr, \\
 W_n^{(2)}(r) &= \frac{\sqrt{2m}}{\hbar} \frac{1}{\chi_n^2(r)} \int^r \chi_n^2(x) \left[ E_n^{(2)} + W_n^{(1)2}(x) - \frac{4ab^4}{6} x^3 \right] dx
 \end{aligned} \tag{19}$$

for any state of interest. The above expressions calculate  $W_n^{(1)}(r)$  and  $W_n^{(2)}(r)$  explicitly from the energy corrections  $E_n^{(1)}$  and  $E_n^{(2)}$ , respectively, which are in turn used to calculate the moderating wave function  $u_n(r)$ .

Thus, through the use of equations (18) and (19), after some lengthy and tedious integrals, we find the zeroth order energy shift and their moderating superpotentials as

$$\begin{aligned}
 E_0^{(1)} &= -\frac{\hbar^4 (\ell + 1)^2 (\ell + 2) (2\ell + 3)}{12am^2} b^3, \\
 E_0^{(2)} &= \frac{\hbar^6 (\ell + 1)^3 (\ell + 2) (2\ell + 3) (2\ell + 5)}{48a^2m^3} b^4 \\
 &\quad - \frac{\hbar^{10} (\ell + 1)^6 (\ell + 2) (2\ell + 3) (8\ell^2 + 37\ell + 43)}{1152a^4m^5} b^6, \\
 W_0^{(1)}(r) &= -\frac{\hbar (\ell + 1) b^3 r}{3\sqrt{2m}} \left[ r - \frac{\hbar^2 (\ell + 1) (\ell + 2)}{2am} \right], \\
 W_0^{(2)}(r) &= -\frac{\hbar b^4 a_3 r}{2\sqrt{2m}} \left\{ b^2 r^3 + a_1 r^2 + a_2 \left[ r + \frac{\hbar^2 (\ell + 1) (\ell + 2)}{2am} \right] \right\} \\
 &\quad - \frac{\hbar (\ell + 1)}{2\sqrt{2ma}} E_0^{(2)},
 \end{aligned} \tag{20}$$

in which

$$\begin{aligned}
 a_1 &= \frac{\hbar^2(\ell + 1)(3\ell + 7)b^2}{2am} - \frac{12am}{\hbar^2(\ell + 1)^2}, \\
 a_2 &= \left[ \frac{\hbar^4(\ell + 1)^2(8\ell^2 + 37\ell + 43)b^2}{8a^2m^2} - \frac{3(2\ell + 5)}{2(\ell + 1)} \right], \\
 a_3 &= \frac{\hbar^2(\ell + 1)^3}{18am}.
 \end{aligned} \tag{21}$$

Therefore, setting  $\beta = b/a$ , the analytical expression for the ground  $s$ -state energy is explicitly given, to order  $\beta^6$ , in atomic units ( $\hbar = m = 1$ ) :

$$E_0/a^2 = -2 + \beta - \frac{1}{2}\beta^3 + \frac{5}{8}\beta^4 - \frac{43}{192}\beta^6 + \dots, \tag{22}$$

and the full radial wavefunction is given by

$$\psi_{n=0,\ell}(r) \approx \chi_{n=0,\ell} \exp\left(-\frac{\sqrt{2m}}{\hbar} \int^r (W_0^{(1)}(x) + W_0^{(2)}(x)) dx\right). \tag{23}$$

Hence, the explicit form of the full wave function in (23) for the ground state is

$$\psi_{n=0,\ell}(r) = \left[ \frac{4ma}{(\ell + 1)\hbar^2} \right]^{\ell+1} \frac{1}{(\ell + 1)^2} \sqrt{\frac{2am}{\hbar^2(2\ell + 1)!}} r^{\ell+1} \exp(P(r)), \tag{24}$$

with  $P(r) = \sum_{i=1}^5 p_i r^i$  is a polynomial of fifth order having the following coefficients:

$$\begin{aligned}
 p_1 &= \frac{(\ell + 1)}{2a} E_0^{(2)} - \frac{2am}{(\ell + 1)\hbar^2}, & p_2 &= \frac{9(\ell + 2)}{4(\ell + 1)^2} a_3^2 a_4 b^4, \\
 p_3 &= \frac{1}{6} a_3 a_4 b^4, & p_4 &= \frac{1}{8} a_1 a_3 b^4, & p_5 &= \frac{1}{10} a_3 b^6,
 \end{aligned} \tag{25}$$

where  $a_4 = a_2 + \frac{12am}{\hbar^2(\ell+1)^2b}$ .

### 3.2. Excited state calculations ( $n \geq 1$ )

The calculations lead to a handy recursion relations in the case of ground states, however it becomes extremely cumbersome in the description of radial excitations when nodes of wavefunctions are taken into account, in particular during the higher order calculations. Although several attempts have been made to bypass this difficulty and improve calculations in dealing with excited states, (cf., e.g. [19], and the references therein) within the frame of SSQM.

Using equations (5) and (17), the superpotential  $W_n(r)$  which is related to the excited states can be readily calculated by means of equations (18) and (19). Hence, the first-order corrections in the first excited state ( $n = 1$ ) are

$$E_1^{(1)} = -\frac{\hbar^4 (\ell + 2)^2 (\ell + 7) (2\ell + 3)}{12am^2} b^3,$$

$$W_1^{(1)}(r) = -\frac{\hbar (\ell + 2) b^3 r}{3\sqrt{2m}} \left[ r + \frac{\hbar^2 (\ell + 2)(\ell + 3)}{2am} \right]. \quad (26)$$

Consequently, substitution of the last equation into equation (19) allows us to write down

$$E_1^{(2)} = \frac{\hbar^6 (\ell + 2)^3 (\ell + 11) (2\ell + 3) (2\ell + 5)}{48a^2 m^3} b^4$$

$$- \frac{\hbar^{10} (\ell + 2)^6 (\ell + 3) (2\ell + 3) (7\ell^2 + 101\ell + 211)}{1152a^4 m^5} b^6. \quad (27)$$

Therefore, the analytical expressions for the first-excited  $s$ -state energy is explicitly given, to order  $\beta^6$ , in atomic units:

$$E_1/a^2 = -\frac{1}{2} + \beta - 7\beta^3 + \frac{55}{2}\beta^4 - \frac{211}{2}\beta^6 + \dots \quad (28)$$

The related radial wavefunction can be expressed in an analytical form by means of equations (18), (19) and (23), if required. The approximation used in this work would not affect considerably the sensitivity of the calculations. On the other hand, it is found analytically that our investigations put forward an interesting hierarchy between  $W_n^{(1)}(r)$  terms of different quantum states in the first order after circumventing the nodal difficulties elegantly,

$$W_n^{(1)}(r) = -\frac{\hbar (n + \ell + 1) b^3 r}{3\sqrt{2m}} \left[ r + \frac{\hbar^2 (n + \ell + 1)(n + \ell + 2)}{2am} \right], \quad (29)$$

which, for the second excited state ( $n = 2$ ) leads to the first-order correction

$$E_2^{(1)} = -\frac{\hbar^4 (\ell + 3)^2 (\ell + 2) (2\ell + 23)}{12am^2} b^3,$$

$$W_2^{(1)}(r) = -\frac{\hbar (\ell + 3) b^3 r}{3\sqrt{2m}} \left[ r + \frac{\hbar^2 (\ell + 3)(\ell + 4)}{2am} \right]. \quad (30)$$



Table 1  
Calculated binding energy eigenvalues for  $0 < \beta \leq 1.0$  up to order  $\beta^6$ .

$\beta$	$E_{00}/a^{2a}$ [10] <sup>b</sup>	$E_{00}/a^2$	Numerical [10]	$E_{10}/a^{2a}$	$E_{10}/a^2$ [10] <sup>b</sup>	Numerical [10]	$E_{20}/a^{2a}$
0.01	-1.9900005			-0.4900067			-0.2122538
0.02	-1.9800039	-1.98000	(-1.98000)	-0.4800516	-0.48005	(-0.48000)	-0.2024526
0.03	-1.9700130			-0.4701668			
0.04	-1.9600304	-1.96003	(-1.96003)	-0.460378	-0.46038	(0.46033)	-0.183718
0.05	-1.9500586	-1.95006		-0.4507047	-0.45070		
0.06	-1.9401161	-1.94010	(-1.94010)	-0.4411605	-0.44116	(0.44115)	-0.1662097
0.07	-1.9301565			-0.4317531			
0.08	-1.9202305	-1.92023	(-1.92023)	-0.4224852	-0.42246	(-0.4221)	-0.1495594
0.09	-1.9103236			-0.4133547			
0.10	-1.9004377	-1.90044	(-1.90044)	-0.4043555	-0.4043	(-0.4048)	-0.1334655
0.20	-1.8030143	-1.803		-0.318752	-0.312		
0.30	-1.7086008	-1.70844	(-1.70958)	-0.2431595		(-0.274)	
0.40	-1.6169173						
0.50	-1.5269368	-1.523	(-1.537)				
0.60	-1.4374490						
0.70	-1.3477860	-1.321	(-1.384)				
0.80	-1.2587093	-1.2					
0.90	-1.1734581						
1.00	-1.0989583	-0.875	(-1.194)				

<sup>a</sup> Calculations to order  $\beta^6$ .

<sup>b</sup> Calculations to order  $\beta^4$ .

Hence, substituting  $W_2^{(1)}(r)$  into equation(19) gives the energy correction in the second-order as

$$E_2^{(2)} = \frac{\hbar^6 (\ell + 2) (\ell + 3)^2 (2\ell + 5) (2\ell^2 + 45\ell + 153)}{48a^2m^3} b^4 - \frac{\hbar^{10} (\ell + 2) (\ell + 3)^5 (16\ell^4 + 474\ell^3 + 3879\ell^2 + 12118\ell + 12873)}{1152a^4m^5} b^6. \tag{31}$$

Therefore, the analytical expressions for the second-excited  $s$ -state energy, to order  $\beta^6$ , in atomic units:

$$E_2/a^2 = -\frac{2}{9} + \beta - \frac{69}{2}\beta^3 + \frac{2295}{8}\beta^4 - \frac{347571}{64}\beta^6 + \dots \tag{32}$$

#### 4. Results And Conclusions

Some numerical values of the perturbed energies of the  $1s$ ,  $2s$  and  $3s$  states, in the atomic units, for various values of  $\beta$  in the range  $0 \leq \beta \leq 1.0$  are

Table 2  
Improvement in energy with respect to orders of  $b/a$ .

$\beta$	$(E_{00}/a^2)_0$	$(E_{00}/a^2)_1$	$(E_{00}/a^2)_3$	$(E_{00}/a^2)_4$	$(E_{00}/a^2)_6$
0.02	-2.0	-1.98	-1.980004	-1.9800039	-1.9800039
0.05	-2.0	-1.95	-1.9500625	-1.9500586	-1.9500586
0.08	-2.0	-1.92	-1.920256	-1.9202304	-1.9202305
0.20	-2.0	-1.80	-1.804	-1.803	-1.8030143
0.50	-2.0	-1.50	-1.5625	-1.5234375	-1.5269368
0.80	-2.0	-1.20	-1.456	-1.20	-1.2587093
$\beta$	$(E_{10}/a^2)_0$	$(E_{10}/a^2)_1$	$(E_{10}/a^2)_3$	$(E_{10}/a^2)_4$	$(E_{10}/a^2)_6$
0.02	-0.5	-0.48	-0.480056	-0.4800516	-0.4800516
0.04	-0.5	-0.46	-0.460448	-0.4603776	-0.460378
0.05	-0.5	-0.45	-0.450875	-0.4507031	-0.4507047
0.06	-0.5	-0.44	-0.441512	-0.4411556	-0.4411605
0.08	-0.5	-0.42	-0.423584	-0.4224576	-0.4224852
0.10	-0.5	-0.40	-0.407	-0.40425	-0.4043555
0.20	-0.5	-0.30	-0.356	-0.312	-0.318752
0.30	-0.5	-0.20	-0.389	-0.16625	-0.2431595
$\beta$	$(E_{20}/a^2)_0$	$(E_{20}/a^2)_1$	$(E_{20}/a^2)_2$	$(E_{20}/a^2)_4$	$(E_{20}/a^2)_6$
0.02	-0.2222222	-0.2022222	-0.2024982	-0.2024523	-0.2024526
0.04	-0.2222222	-0.1822222	-0.1844302	-0.1836958	-0.1837180
0.06	-0.2222222	-0.1622222	-0.1696742	-0.1659563	-0.1662097
0.08	-0.2222222	-0.1422222	-0.1598862	-0.1481358	-0.1495594
0.10	-0.2222222	-0.1222222	-0.1567222	-0.1280347	-0.1334655

presented in table 1. Our results are consistent to order  $\beta^6$  with earlier results obtained by applying the large  $N$ -expansion method [10]. Therefore, our results are found in high agreement for small values of  $\beta$  with those given in [10]. On the other hand, for large screening parameter values, the accuracy of our results exceeds the ones given before in [10]. These results also appear to be in close agreement with the results obtained by solving Schrödinger equation numerically with the same potential via Numerov’s method [10]. However, these results tend to deviate slightly as  $\beta$  approaches 1.0 with the results obtained from the numerical solution of Schrödinger equation. Moreover, we illustrate the improvement of energy with respect to orders of  $\beta$  in table 2.

In this work, we have shown that the bound-state energies of a MGESC potential for all eigenstates can be accurately determined within the framework of a novel treatment. Avoiding the disadvantages of the standard non-relativistic perturbation theories, the obtained expressions in the present work have much simple forms than the ones shown in the previous work [10].

Finally, the application of the present technique to MGESC potential for the first time is really of great interest providing leading to analytical expressions for both energy eigenvalues and wave functions which are likely of much interest in different field of physics.

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## References

- [1] P. Anderson, Phys. Rev. 86 (1952) 694; R. Kubo, Phys. Rev. 87 (1952) 568.
- [2] R. A. Ferrell and D. J. Scalapino, Phys. Rev. A 9 (1974) 846; A. J. Bray, J. Phys. A 7 (1974) 2144.
- [3] V. L. Bonch-Bruевич and V. B. Glasko, Sov. Phys. Dokl. 4 (1959) 147.
- [4] N. Takimoto, J. Phys. Soc. Jpn. 14 (1959) 1142.
- [5] V. L. Bonch-Bruевич and Sh. M. Kogan, Sov. Phys. Solid State 1 (1960) 1118; C. Weisbuch and B. Vinter, *Quantum Semiconductor Heterostructures* (Academic, New York, 1993); P. Harrison, *Quantum Wells, Wires and Dots* (Wiley, England, 2000).
- [6] E. P. Prokopev, Sov. Phys. Solid State 9 (1967) 993.
- [7] R. Dutt and Y. P. Varshni, Z. Phys. D 2 (1986) 207.
- [8] R. Dutt, A. Ray and P. P. Ray, Phys. Lett. A 83 (1981) 65; C. S. Lam and Y. P. Varshni, Phys. Rev. A 4 (1971) 1875; D. Singh and Y. P. Varshni, Phys. Rev. A 28 (1983) 2606.
- [9] C.C. Gerry and J. Laub, Phys. Rev. A 30 (1984) 1229.
- [10] R. Sever and C. Tezcan, Phys. Rev. A 36 (1987) 1045.
- [11] B. Gönül, K. Köksal and E. Bakır, Phys. Scr. 73 (2006) 279; B. Gönül, Chin. Phys. Lett. 21 (2004) 1685; B. Gönül, Chin. Phys. Lett. 21 (2004) 2330; B. Gönül and M. Koçak, Mod. Phys. Lett. A 20 (2005) 355; B. Gönül, N. Çelik and E. Olğar, Mod. Phys. Lett. A 20 (2005) 1683; B. Gönül and M. Koçak, Chin. Phys. Lett. 20 (2005) 2742; *ibid.* Mod. Phys. Lett. A 20 (2005) 1983.
- [12] S.M. Ikhdair and R. Sever, arXiv:quant-ph/0604073 [in press, J. Math. Chem. (2006)]; *ibid.* Int. J. Mod. Phys. A21 (2006) 6465; *ibid.* [in press, J. Molec. Struct.: THEOCHEM (2007) doi: 10.1016/j.theochem.2007.01.019]; *ibid.* J. Molec. Struct.: THEOCHEM 806 (2007) 155.
- [13] F. Cooper, A. Khare and U. P. Sukhatme, Phys. Rep. 251 (1995) 267.
- [14] M. Znojil, J. Math. Chem. 26 (1999) 157.
- [15] M. Alberg and L. Wilets, Phys. Lett. A 286 (2001) 7.
- [16] D. J. Doren and D. R. Herschbach, Phys. Rev. A 34 (1986) 2665.
- [17] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer, Berlin, 1957).
- [18] L. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965).
- [19] C. Lee, Phys. Lett. A 267 (2000) 101.